Analytic Derivation & Topographic Application of the Crescent Visibility Parabola Function

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Abstract: **Lunar crescent visibility maps play an importing role in determining the Earth's regions where the very young Moon can be spotted in the evening, thereby supporting aided or unaided observations and establish a lunar calendar. Construction of such a map requires heavy computing, since the visibility function based on the selected criteria must be evaluated for each latitude, longitude and time. Especially the preparation of high-resolution maps is very time consuming; execution times exceeding a minute are common. This paper proposes an analytic approach to solve the separator parabola function to achieve a calculation time of less than one second. Furthermore, the dependence of the lunar visibility on the site height above the sea level has been exercised; the generation of a topographic visibility map and the conformance of such an application to the existing criteria and observations have been studied. Benchmark among the available map software and the developed demonstration tool is included.**

Keywords: lunar crescent visibility map; visibility separator parabola; arc of vision; crescent width; moon visibility criterion; lunar date line; moon elongation

1. INTRODUCTION

It is a well-known phenomenon that stars are not visible during the daytime. The reason is the existence of the atmosphere, which scatters the incoming Sun rays, thence causing a blue sky. The brightness of the sky during the day period is higher than that of the stars, inhibiting their visibility. Whereas the sky brightness caused by scattering also affects the Moon, its brightness exceeds that of the sky such that it can be observed in daytime. However, unlike the stars, the Moon is not a point source and its illuminated width changes periodically. When it is thinner than ca. 5' (arc-minutes), its apparent brightness starts to diminish; after about 3' it will no more be resolvable with naked-eye before the Sun sets. The brightness of a thin crescent can therefore be sufficiently greater only than that of a twilit sky, namely at dusk for the waxing period (new moon) and at dawn for the waning (old moon).

Although the waxing and waning crescents have analogous but opposite behavior, the waxing phase have been more important because of the religious new lunar month determination, which necessitates the perception of the crescent on the western sky. The visibility depends on both the lunar apparent brightness and the darkness of the sky at the Moon's elevation; it can be expressed as the function $V = f(d, b)$, where d and b represent the sky darkness and Moon brightness, respectively. Recent researchers use the parameter ARCV (Arc of Vision) for d, and W (crescent width) for b, such that $V = f(\text{ARCV}, \text{W})$. Yallop [1] and Odeh[1] [2] defined the visibility function at the "best time" as[2]:

$$V = ARCV + 6.3226*W - 0.7319*W^2 + 0.1018*W^3 - k$$

Here, ARCV is in degrees and W in arc-minutes. Özlem argued that Sun and Moon elevations should be processed independently in order to calculate the darkness correctly throughout the whole possible elevation range [3]. So he split ARCV into M and S, namely the (topocentric) Moon and the Sun elevation angles, respectively; where ARCV ≈ M – S + 1. He then defined the visibility at the sea level as:

$$V = -0.28/\tan(M+1.5) - S + 6*\sqrt{W} - 4.9$$

Here the darkness parameter d becomes nonlinear. Nevertheless, since he defined the "best time" of visibility as when M = 2.5, then ARCV = 2.5 – S + 1 and thus his criterion can be expressed as:

$$V = ARCV + 6*\sqrt{W} - k$$

Here k depends on the sight height above the sea level, since he included a correction term to compensate for the higher darkness at increased elevation[3].

For the determination of any lunar month, it is important to know *when*, *where* and *how* the maiden crescent becomes visible. The lunar visibility depends both on time t and position P (latitude φ and longitude ℓ), such that we may define the visibility in another way as $V = f(P, t)$, where $P = \{\varphi;\ \ell\}$. The crescent width W is a function of time only[4],

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[1] Odeh prefers the topocentric arc of vision, which is nearly 1° less than ARCV due to the Moon's parallax, which should be accounted for.

[2] Constant term k is 11.8371 for Yallop and 8.1651 for Odeh.

[3] For best visibility, $k = 0.28/\tan(2.5+1.5)+2.5+1+4.9 \approx 12.4$ at sea level, 11.4 at 1,000m altitude and 10.4 at 4,000m above sea level.

[4] The topocentric width does depend on the position, or rather on M; however, the change is neglected since M is nearly constant at best visibility.

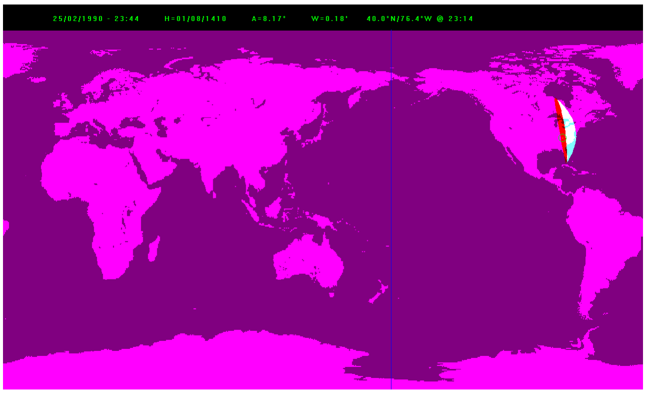
whereas ARCV varies with both time and position. The threshold for visibility V_0 changes according to the method of observation. The unaided (naked-eye) visibility limit will be higher than the aided (with binocular or telescope), that is a higher V_0 value is necessary for the naked-eye observation. In general, the condition $V(P, t) \geq V_0$ must be satisfied for the lunar visibility.

2. CRESCENT VISIBILITY MAP

The definition for the beginning of a religious month is the physical observation of the new crescent on the west horizon after sunset, upon which the 1st day of the new lunar month starts. The moon-sighting may be divided into three categories, namely local, regional or global [4]. Local sighting requires that the crescent is actually witnessed on the district; every region is liable to its own observation and the start date of a month may differ among the countries. Regional sighting, on the other hand, accepts that if authentic moon-sighting news comes from neighbor areas, then local sighting is no more determinative. The positive witness information should come until the beginning of the local isha time or, at most, until its preferred end (one third of the night or midnight). The global sighting approach is an extended interpretation of this thought; if the new crescent has been attested in any location of the world, then the entire world will ratify the start of the month. Under this admission, an International Hijri Calendar for the unification of fasting and pilgrimage could be established [5].

Whether to rely on the local, regional or rather global moon-sighting, it will be necessary to know at which places on Earth the maiden crescent becomes detectable. The preferred method is to prepare a world map, generally with Mercator projection, showing the areas where the crescent is visible. The map is generally constructed for 24 hours following the first appearance; but extended (two-day) maps are also common. The production of a visibility map involves lengthy and tedious calculations; the visibility should be computed for every pixel of the map. Since $V = f(P, t)$, the visibility function is to be evaluated by scanning all the three dimensions, i.e. the latitude, the longitude and the time. Any visibility map software therefore needs many seconds, sometimes several minutes to complete the map, depending on the required resolution. Some benchmark results are presented in the *Conclusion* section.

Within the computation algorithm, if the time is incremented as the highest rank, then the time-dependant parameters like the crescent width, Earth/Moon declination and right ascension values are computed once for each time increment. Time interval depends on the resolution of the map; since the earth rotates 0.25° every minute, it will be a proper choice to select the time slice as 1 minute for a 1440x720 map. For a 720x360 map in contrast, an interval of 2 minutes should suffice. Note that the computing effort is therefore proportional to the cube of the resolution; i.e. an 8-fold calculation time will be necessary for the former map. The next step will be to scan in the latitude & longitude, and check whether the output of the visibility function exceeds the set threshold. The confined area with $V \geq V_0$ shows then the instantaneous visibility for any specific time t. If the Moon brightness is under a certain limit, i.e. $W < W_0$, the crescent is not visible on any point of the world, so the map will be empty. As the time is incremented, the visibility first starts on a single point whenever $W = W_0$ at time t_0 and then the area grows vertically and westward, like a "Big Bang". Figure 1 shows the visibility maps for Sha'ban 1410, obtained by the tool *Ehille* (p = 30%, H = 1,500m). The global visibility commences on February 25th, 1990 at 23:14 GMT on the location P_0 = {40.0°N; 76.4°W}. The crescent remains visible ($V > V_0$) within the red area only and it is not visible ($V < V_0$) outside. Hence, on the contours of the red-painted region, $V = V_0$. The upper map is an earlier snapshot, just 30 minutes after t_0, whereas the lower map is taken after 5.5 hours.



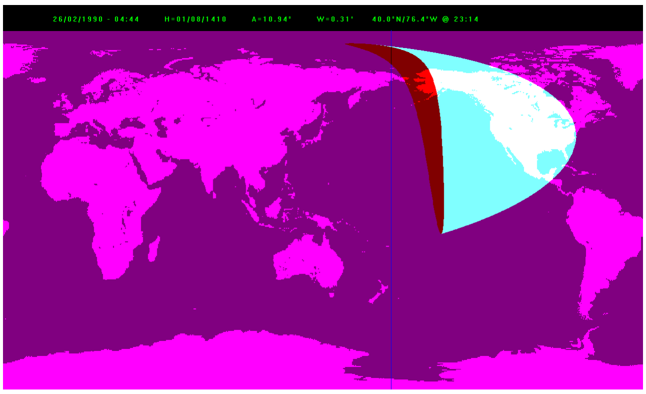


Figure 1 – Instantaneous Visibility Maps for Sha'ban 1410

The tiny white dot (encircled in green) within the red region on the upper map is the location of observation

(35.6°N / 83.5°W) by John Pierce, who still owns the world record for naked-eye observation. Note that the horizontal width at that point is rather tight and the crescent was visible for ca. 15 minutes only.

The final visibility map is constructed by cumulating all the instantaneous areas, as shown in white/cyan in Figure 1. The shape resembles a parabola, with the apex on the point of first visibility P_0, broadening to west. The locations inside this pseudo-parabola start the lunar month on that evening and those outside wait for another day, if they adopt local moon-sighting. For regional sighting, locations outside but near the parabola may also start the new month. In case of global sighting, the whole world should start the month if the crescent becomes visible on the American west coast (generally before midnight GMT); otherwise the following day is the 1st. However, on some of the far-eastern countries, dawn may have already started at the time the crescent is just visible on the west-coast of the America, so it is also logical to divide the world into several regions [6].

Different threshold values have been selected by the researchers in order to specify various observation conditions. Yallop preferred the following categories along with the associated V_0 values[5]:

(A) Easily visible:	+2.16
(B) Visible under perfect conditions:	−0.14
(C) May need optical aid:	−1.60
(D) Will need optical aid:	−2.32
(E) Not visible with a telescope: above Danjon limit	−2.93

Odeh, in contrast, divided the visibility range into three zones, by combining Yallop's (B) and (C) zones into a single one (B):

(A) Visible by naked eyes	+5.65
(B) Visible by optical aid: could be seen by naked eyes	+2
(C) Visible by optical aid only:	−0.96

Note that the constant offset values (k) are different for Yallop and Odeh by 3.672, such that the naked-eye thresholds differ in fact just by 0.072. Similarly, the first threshold (A) differs only by 0.182. However, Odeh widened the area visible by optical aid, considering some recent observations, by 2.312. Özlem, on the other hand, stated his formula for the border of 50% probability with the naked-eye [3], which corresponds to $V_0 = 0$ (Yallop). He didn't make any clarification about the optically aided visibility.

[5] Here, Yallop's threshold values are multiplied by 10 since they were divided by ten in their original equations.

Below are presented sample visibility maps showing multiple zones in various colors. Figure 2 (top) is a 0.25° resolution map prepared by using Yallop's criterion; whereas Figure 2 (bottom) is a 2° map produced with the software *Accurate Times* by Odeh. Note that the limit for unaided visibility is nearly identical for both maps (compare cyan/yellow border on the upper map with purple/blue on the lower), but the aided one is extended by Odeh (compare yellow on the upper with blue on the lower). Besides, the first threshold (A) is very similar (compare both green areas on the most-left side), as explained above.

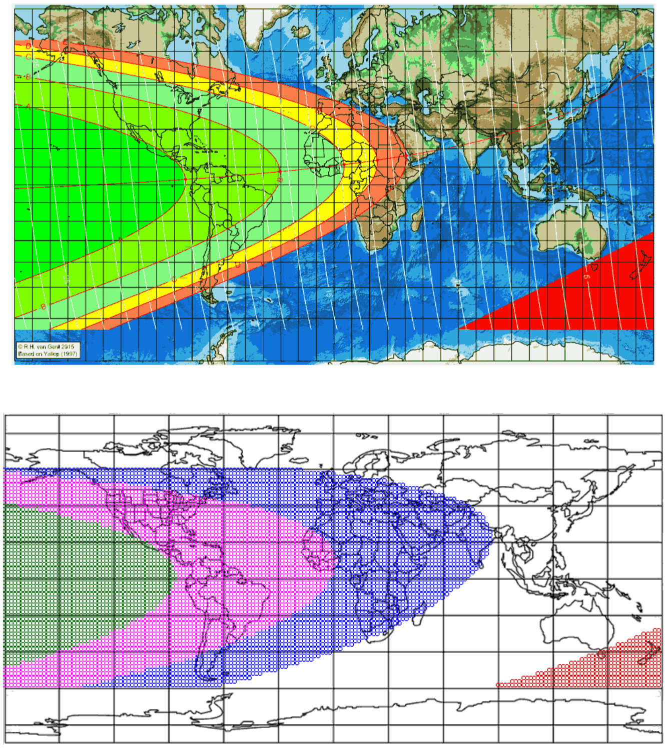


Figure 2 – Visibility Maps for Safar 1440

3. POINT of BEST VISIBILITY

The unique position P_0 for a specific threshold V_0 is the initial location where the moon becomes visible at t_0. This point is the apex of the associated parabola, where the visibility is at maximum. So the apex of each succeeding parabola with threshold V_n is actually the best position P_n (with the highest global visibility) on the Earth for that specific time t_n. If we combine all these points for each incremented time t_n, we end up with a Best Visibility Line (horizontal red line in Figure 2, top). This line is the quasi-mid line of every parabola on the map. We will define the functions $P_b(t)$ and $V_b(t)$, which express the best visibility position and the visibility value at that position, respectively.

The position consists of the best latitude and the best longitude such that $P_b(t) = \{\varphi_b(t); \ell_b(t)\}$.

The best visibility value & position can be obtained by taking the partial derivative of the visibility function, $\partial V/\partial P$. The maximum (best) visibility value will be reached when the partial derivative becomes zero at a certain position. Remembering that $V = f(\text{ARCV, W})$ and also it was former stated that not the crescent width but the Arc of Vision depends on position, the best visibility should occur when ARCV is maximum. Using spherical orthogonality, ARCV can be expressed as:

$$\text{ARCV} = \text{acos}(\cos(\text{ARCL}) / \cos(\text{DAZ}))$$

ARCL (Arc of Light), which is actually the Sun-Moon elongation, does not depend on position, thus ARCV will be maximum when the only position-dependent parameter DAZ (delta azimuth) is zero; ARCV equals to ARCL in that situation and the best visibility can be found by replacing ARCV by ARCL in the visibility equations. For the criterion by Özlem, as example:

$$V_b(t) = \text{ARCL} + 6*\sqrt{W} - k$$

The global visibility starts when $V_b(t) \geq V_0$, so it is superfluous to scan the latitude & longitude dimensions before this condition is met.

On the best global visibility position $P_b(t)$, DAZ will be zero, so the Moon and the Sun are positioned strictly vertical, as shown in Figure 3.

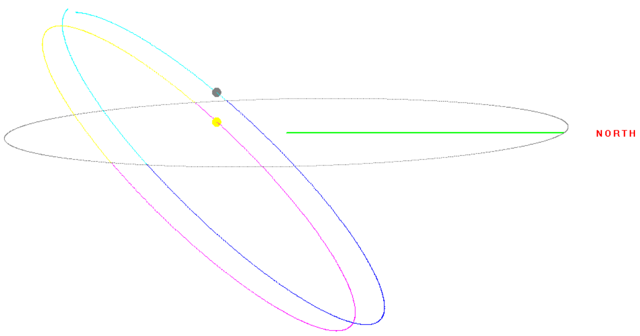


Figure 3 – Best Position Trajectories

The general azimuth/elevation formula, using spherical trigonometry, is:

$$\sin(\delta) = \cos(a)*\cos(e)*\cos(\varphi)+\sin(e)*\sin(\varphi)$$

Here, δ stands for declination, a shows the azimuth (horizontal angle) and e denotes the elevation (vertical angle). We can write this equation both for the Moon and the Sun:

$$\sin(\delta_m) = \cos(a_m)*\cos(e_m)*\cos(\varphi)+\sin(e_m)*\sin(\varphi)$$

$$\sin(\delta_s) = \cos(a_s)*\cos(e_s)*\cos(\varphi)+\sin(e_s)*\sin(\varphi)$$

$$\text{DAZ} = a_m - a_s$$

$$\text{ARCV} = e_m - e_s$$

Since the difference of the Moon and Sun vertical angles equals to ARCV, they can be individually calculated, using the "best time" rules. For Yallop / Odeh, $4*e_m = -5*e_s$ and for Özlem, $e_m \approx 3.5$, where e_m and e_s represent the (geocentric) Moon and Sun elevations, respectively. For the special case of best visibility, where ARCV = ARCL and DAZ = 0:

$$a_m = a_s$$

$$e_m - e_s = \text{ARCL}$$

To obtain the best visibility latitude φ_b, we equal the Moon and the Sun azimuth using the azimuth/elevation formula:

$$\text{acos}((\sin(\delta_m)-\sin(e_m)*\sin(\varphi_b))/\cos(e_m)/\cos(\varphi_b)) =$$
$$\text{acos}((\sin(\delta_s)-\sin(e_s)*\sin(\varphi_b))/\cos(e_s)/\cos(\varphi_b))$$

After truncation:

$$(\sin(\delta_m)-\sin(e_m)*\sin(\varphi_b))/\cos(e_m) =$$
$$(\sin(\delta_s)-\sin(e_s)*\sin(\varphi_b))/\cos(e_s)$$

$$\sin(\varphi_b) = (\sin(\delta_m)*\cos(e_s)-\sin(\delta_s)*\cos(e_m)) / (\sin(e_m)*\cos(e_s)-\sin(e_s)*\cos(e_m))$$

The denominator in the above formula is nothing but *sin*(ARCV) which is equal to *sin*(ARCL) here, so the final equation becomes:

$$\sin(\varphi_b) = (\sin(\delta_m)*\cos(e_s)-\sin(\delta_s)*\cos(e_m))/\sin(\text{ARCL})$$

The best latitude φ_b for any given time t is hence obtained by this simple formula:

$$\boldsymbol{\varphi_b = \text{asin}((\sin(\delta_m)*\cos(e_s)-\sin(\delta_s)*\cos(e_m))/\sin(\text{ARCL}))}$$

If we look closer into Figure 3, we can see that an orthogonal triangle can be formed with the Sun-Moon line as hypotenuse. This triangle is sketched in Figure 4, with the red side parallel to the ecliptic and the blue side vertical to the ecliptic. The lower angle is then the best latitude φ_b. Note that the blue side is the difference of the Moon & Sun declinations ($\delta_m - \delta_s$), and the red side is the difference of their right ascensions ($\alpha_m - \alpha_s$). This is a spherical triangle where the sides are circular arcs. If it were a plain triangle with linear sides, then we could calculate the best latitude (in radians) simply by dividing $\Delta\delta$ by ARCL (sine-rule). In fact, also the above formula implies this fact when we replace $cos(d\gamma) = 1$ and $sin(d\gamma) = d\gamma$, since a spherical triangle with infinitesimal angles transforms into a plain one.

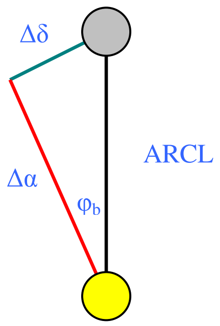


Figure 4 – The Orthogonal Triangle

The following equations are applicable to find the associated best longitude ℓ_b:

$$\ell = \omega - GMST + \alpha$$

$$\sin(e) = \cos(\omega)*\cos(\delta)*\cos(\varphi)+\sin(\delta)*\sin(\varphi)$$

Here, GMST is the Greenwich Mean Sidereal Time in degrees, ω shows the local hour-angle, which should be negated for the case of a waning crescent, and α stands for the right-ascension. As such, ℓ_b can be obtained by:

$$\boldsymbol{\ell_b = \pm acos((\sin(e_m)-\sin(\delta_m)*\sin(\varphi_b))/\cos(\delta_m)/\cos(\varphi_b)) - GMST + \alpha_m}$$

Of course the Sun parameters e_s, δ_s and α_s may be used instead and the following equation is valid as well:

$$\ell_b = \pm acos((\sin(e_s)-\sin(\delta_s)*\sin(\varphi_b))/\cos(\delta_s)/\cos(\varphi_b)) - GMST + \alpha_s$$

Calculation of GMST, α_s, δ_s, α_m, δ_m, ARCL and W for any time t is given in the *Appendix* section.

4. PARABOLA FUNCTION

We denoted the time of the first visibility as t_0, at which time $V = V_0$ and the crescent with W_0 is only visible on a single point P_0, namely the best visibility point for that threshold. As the time passes over, the crescent gets thicker ($W > W_0$) and the Moon becomes visible on a small vertical band (red zone in Figure 1) around the Best Visibility Line. The visibility drops as moved from the mid-line up and down to the edges of the parabola; on the parabolic line $V = V_0$ and at the outside $V<V_0$. DAZ on the other hand, which is zero on the mid-line, will increase as moved away from the line. For a specific time $t_n > t_0$, the maximum DAZ within the visible region, denoted as DAZ_n, will be reached at the visibility margin (on the parabolic line). Because ARCV is orthogonal to DAZ, it will be at minimum on that visibility border and it is denoted as $ARCV_n$. If the mid-line (maximum) visibility at t_n is labeled as V_n, then the visibility difference ΔV between the mid-line and the contour will be equal to the increase of the best visibility from the time t_0 to t_n, which can be written as:

$$\Delta V = V_n - V_0$$

But since the visibility V at a certain time is directly proportional to ARCV because W is constant (see the visibility functions described by Yallop, Odeh and Özlem), $\Delta V = \Delta ARCV$. So the Arc of Vision on the contour must be ΔV smaller than that on the Best Visibility Line[6], which is ARCL. Therefore we may find DAZ_n:

$$\Delta V = V_n - V_0 = ARCL_n - ARCV_n$$

$$ARCV_n = ARCL_n - \Delta V = ARCL_n - (V_n - V_0)$$

$$DAZ_n = acos(\cos(ARCL_n) / \cos(ARCV_n))$$

Since DAZ_n is the difference of the azimuths between the Moon and the Sun, the corresponding latitude where this DAZ appears could be found using the general azimuth/elevation formula:

$$DAZ = acos((\sin(\delta_m)-\sin(e_m)*\sin(\varphi))/\cos(e_m)/\cos(\varphi)) - acos((\sin(\delta_s)-\sin(e_s)*\sin(\varphi))/\cos(e_s)/\cos(\varphi))$$

Unfortunately, the former truncation is no more possible in this case and the extraction of the latitude becomes much strived. Instead, we will try to solve the problem using the spherical orthogonal triangle (green-painted in Figure 5) with the vertical sides DAZ / ARCV and the hypotenuse ARCL. The angle opposite to DAZ is the tilt angle θ, which is formed by rotating our preceding $\Delta\delta$ / $\Delta\alpha$ triangle in Figure 4 as to reach the necessary $ARCV_n$. Therefore we may write:

$$\sin(\theta) = \sin(ARCV) / \sin(ARCL)$$

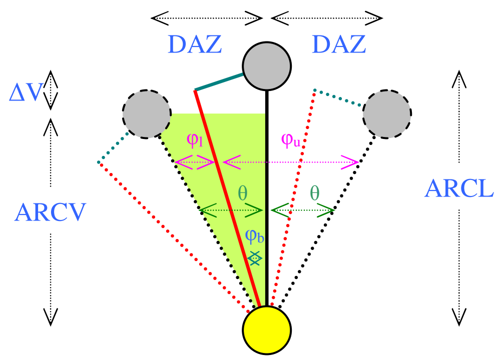


Figure 5 – Rotation of the Orthogonal Triangle

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[6] For Yallop/Odeh: $ARCV_n = -6.3226*W+0.7319*W^2-0.1018*W^3+k+V_0$
For Özlem: $ARCV_n = -6*\sqrt{W} + k + V_0$

Then the upper (φ_u) and the lower (φ_l) latitudes for the contour points of the parabola will therefore be:

$$\varphi_u = \varphi_b + \theta$$
$$\varphi_l = \varphi_b - \theta$$

However, this assumption is only valid for plain triangles, not for spherical ones, so it may produce consistent results only for small angle values. The actual latitude values will hence deviate from the calculated, especially on large values of the angles δ, ω and θ. We will thus try to fit an empirical correction factor c to get coherent results:

$$\varphi_u = \varphi_b + \theta * c_u$$
$$\varphi_l = \varphi_b - \theta * c_l$$

The following coefficients produce latitude values with an error of typically < 1° for polar zones and < 0.5° otherwise:

$$\mathbf{c_u = \cos(|\delta_m|+0.25*\cos^2(\omega_{m,u}))}$$
$$\mathbf{c_l = \cos(|\delta_m|+0.25*\cos^2(\omega_{m,l}))}$$

The hour-angle values for the upper ($\omega_{m,u}$) and the lower ($\omega_{m,l}$) points depend on the latitudes φ_u and φ_l, respectively, which provokes recursion. But the time increments are sufficiently small during the build-up of the parabola such that the hour-angles change very little in-between. So we may use here the hour-angle of the previous sample, which has already been computed to find the longitude value. This technique significantly reduces the execution time. The next calculated sample of φ_u may become lower than (or φ_l may become higher than) the previous sample because of the change in the hour-angle, which should be inhibited by the limiting *min*/*max* functions to prevent oscillations.

If a higher accuracy is necessitated, the latitude values obtained hereby should be inserted back into the DAZ equation and then fine-tuned in a recursive manner; however this measure will increase the amount of computation. Additionally, the initial time increments near the apex may be kept smaller to produce a smoother curve, since the parabola widens very fast at start.

On the polar zones, the Sun and/or the Moon might not reach the necessary elevation as required by the best-time rule. At the poles, the Sun & Moon trajectory becomes horizontal, which can be visualized by tilting the orthogonal triangle in Figure 4 such that the red side becomes horizontal and the blue side vertical. In this case $e_m = \delta_m$ and $e_s = \delta_s$ for the North Pole ($e_m = -\delta_m$ and $e_s = -\delta_s$ for the South Pole) thus ARCV = $\Delta\delta$. Hence a crescent may only be visible at the North Pole, when $\delta_m > 0$, $\delta_s < 0$ (opposite for the South Pole) and $\Delta\delta$ sufficiently big. Criteria incorporating solely the Arc of Vision as the sky illumination parameter produce erroneous results within the polar region; generated parabolas usually include one of the poles, therefore the parabolas are cut near the polar zone (Figure 2). Indeed on the 1st of Safar 1440 for example, the Sun & Moon are both over the horizon at the South Pole, nearly at 7° elevation, such that it is impossible for the crescent to be distinguished. *Extended Crescent Visibility Criterion* on the other hand, uses separated M and S parameters, whereby it becomes still possible to define visibility at those extreme regions. The latitude values computed analytically using the above equations, however, will deviate from the criterion. Also here, the preferred correction method will be to find an empirical approximation instead of pixel-by-pixel calculation, in order to speed up the process. Following equations are applied for the cases where the expected Moon elevation is not achievable, which is characterized by $|cos(\omega_m)| > 1$:

$$\varphi_u' = 0.15*\varphi_u+0.85*\mathrm{asin}(\sin(\delta_m+90°-e_m)) \quad » \quad \cos(\omega_{m,u})>0$$
$$\varphi_u' = 0.35*\varphi_u+0.65*\mathrm{asin}(\sin(\delta_m+90°-e_m)) \quad » \quad \cos(\omega_{m,u})<0$$
$$\varphi_l' = 0.15*\varphi_l+0.85*\mathrm{asin}(\sin(\delta_m-90°+e_m)) \quad » \quad \cos(\omega_{m,l})>0$$
$$\varphi_l' = 0.35*\varphi_l+0.65*\mathrm{asin}(\sin(\delta_m-90°+e_m)) \quad » \quad \cos(\omega_{m,l})<0$$

Note that the weighing coefficients are different for whether $cos(\omega_m)$ is positive or negative (i.e. the Moon is below the limiting elevation or above), basically since the impact of the Moon elevation to the visibility is not linear. Here again, *min*/*max* functions for limiting should be inserted to prohibit that φ_u' becomes lower than (or φ_l' becomes higher than) the previous sample.

One example for visibility at the pole could be the 1st of Shawwal 1430; the crescent becomes visible at the South Pole when W = 0.8' (δ_m = –10.5 and δ_s = 1) as per the *Extended Crescent Visibility Criterion*. Anyway, the visibility at the pole has not been attested by observation yet.

5. DEMONSTRATION TOOL

In contrast to the Windows application *Ehille*, which was produced by Özlem to demonstrate the *Extended Crescent Visibility Criterion* [3]; a WEB tool, *Urcun*, is designed to test the performance of the analytical approach explained herein. *Urcun* is written in HTML/JavaScript in order to achieve portability and OS independence. The date can be entered either as Gregorian or as Hijri. It is based on the same criterion by Özlem and the altitude is hence adjustable to see the effect of height of the sight above the sea level. *Urcun* can display the first visibility of a waxing crescent (start of lunation) as well as the last visibility of a waning (end of lunation), as shown in Figures 6 and 8, respectively. Either map can further be extended to view the 2nd day; i.e. the next day for the start of the month, or the previous day for the end (Figure 7).

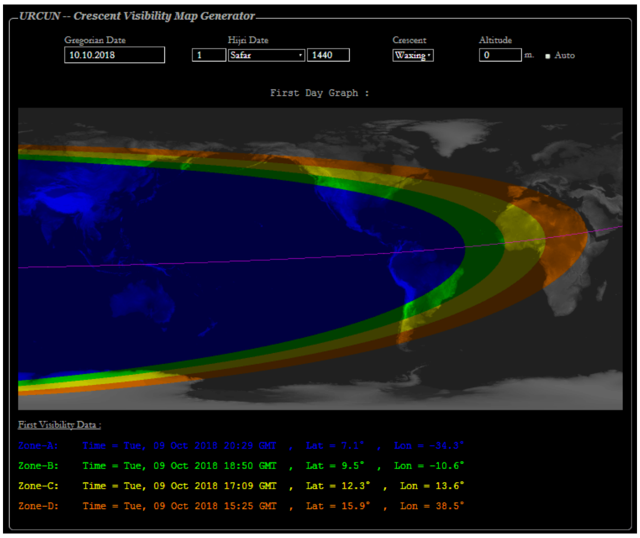

Figure 6 – First Visibility Map for Safar 1440 by Urcun

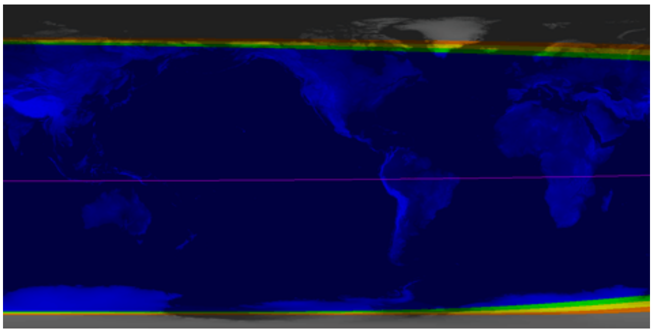

Figure 7 – 2nd Day Map for Safar 1440 by Urcun

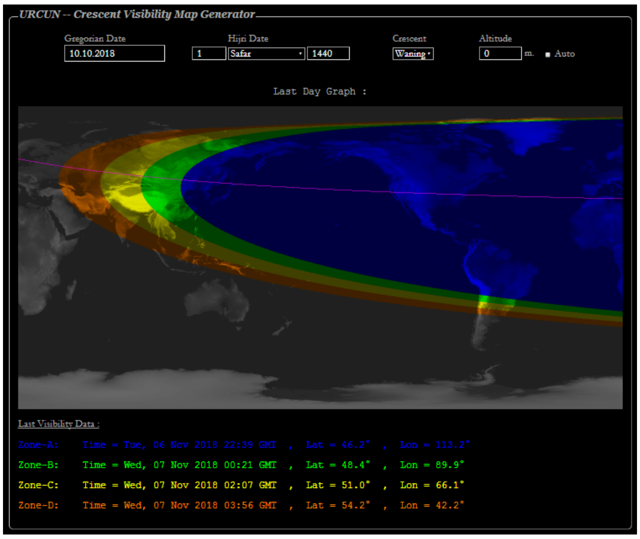

Figure 8 – Last Visibility Map for Safar 1440 by Urcun

Four distinct zones are defined for *Urcun*, which are uniquely colored on the map. The best time & position for each zone (apex of each parabola) is printed below the map. The visibility condition and the associated threshold value (V_0) for each zone are defined as follows:

(A) Easily visible even by naked-eye:	+1
(B) Usually visible by naked-eye: easily visible by optical aid	0
(C) Rarely visible by naked-eye: usually visible by optical aid	−1
(D) Not visible by naked-eye: rarely visible by optical aid	−2

These conditions are valid for a clear sky (visibility range > 20 km) with an experienced observer having good visual acuity. Existence of haze will adversely affect the visibility; the value may decrease by 1 or 2. The presence of city lights is also evaluated to worsen the visibility by increasing the sky illumination [12]. Besides are the seasonal variations an important factor; Schaefer computes that the critical visibility decreases by 0.6 in June and increases by 0.8 in December, as compared to equinoxes [13].

Unlike other criteria, the visibility thresholds V_0 for *Urcun* are chosen to be simple integers, such that the visibility of each zone differs from its neighbors by $\Delta V = 1$. The reason is that the border of zones A-B corresponds to 100% probability of naked-eye visibility, whereas the border of zones B-C corresponds to 50% and the border of zones C-D to 0%. V_0 is set to zero for 50% probability and the spread is taken as $\sigma = \pm 1$ per the *Extended Crescent Visibility Criterion* [3].

The amount of increase in visibility through the use of an optical aid is not agreed among the researchers. Several recent observations are considered to be credible by some authorities, whereas some others find them doubtful. Here, we prefer to make a simple photometric analysis to estimate this visibility gain:

For a moderate crescent width value of W = 0.5’, the necessary Sun depression is calculated as 3.61° [3]. The corresponding background sky brightness L_B will be 2.2×10^6 nL [17]. The associated Blackwell Contrast Threshold C_{th} is then around 2.5 [18]. The contrast is defined as:

$$C = (L_m - L_B) / L_B$$

Accordingly, the Moon brightness is found as $L_m = 7.7 \times 10^6$ nL. Under optical magnification, the Blackwell Contrast Threshold will approach to zero, such that the crescent could be seen when the background brightness reaches almost the Moon brightness. The sky brightness of $L_B = 7.7 \times 10^6$ nL occurs when the Sun depression is nearly 2.5° [17]. This corresponds to 1.11° decrease in ARCV. Note that this drop will shrink W and thereby reduce the visibility back such that the effective drop will be less. The gain will be somewhat higher for thinner crescents and lower for thicker crescents. Thus, for *Urcun*, the 50% probability threshold for optical aided observations has been set at $V_0 = -1$, which is 1° lower than the visibility threshold for the naked eye.

The simplified algorithm of *Urcun* is depicted below:

At start, the tool finds the first/last day of the lunation and calculates the elongation at Greenwich midnight. The initial scan time is adjusted to approx. 2 hours before the first (or after the last) visibility for the outer zone. The Earth map is scrolled such that the first best visibility sample is located at the border. It then enters the time loop. For each incremented time, ARCL and W are computed and the visibility difference ΔV is found. Note that the time is decremented for the case of a waning crescent. φ_b and ℓ_b are calculated next and the associated pixel is marked as to draw the Best Visibility Line. A negative ΔV value tells that the global visibility has not been achieved yet so the loop is continued until ΔV becomes positive. At the point of visibility onset, corresponding values of t, φ_b and ℓ_b are displayed. The tilt angle θ, the upper/lower latitudes φ_u/φ_l and longitudes ℓ_u/ℓ_l are calculated thereafter. The associated pixels are then marked and adjoined as to form the parabola. The calculations are repeated for each of the four zones. Logically, the number of time increments should be equal to the horizontal pixels of the map. However, the contours of the parabola move slower as they approach the poles, so the algorithm increments the time 2,000 times for the 1,440 column map and the locations outside the map are cropped. After the time loop has been completed, the painting routine starts. Here, it scans the map horizontally line-by-line, from right to left for the waxing and from to left right for the waning crescent. Color is changed every time a parabola contour has been crossed. The brightness of every pixel is adjusted to be nearly proportional to the square-root of the altitude at that location.

6. TOPOGRAPHIC APPLICATION

Dependence of the lunar visibility on the altitude has been mentioned in the literature [1, 8, 9, 10, 15], which was formulated by Özlem [3] and verified by the author after several land-based and airborne observations. Although *Urcun* (and its predecessor *Ehille*) incorporate altitude correction, the general effect of selecting a higher site height is the eastward shift of the parabolas. Nevertheless, the Earth surface is not uniform in altitude; therefore the assumption that "all terrain within a uniquely colored zone has the same visibility" is not true. It will be more realistic to calculate the visibility of each location (pixel) regarding its specific altitude value. Hence, *Urcun* includes an "auto-altitude" mode, where a topographic visibility map is prepared. On this map, the visibility of each pixel is altitude-corrected and then painted accordingly (Figure 9). In order not to inflate the execution time and thus upset the analytical derivation feature, the visibility is not calculated pixel-by-pixel. Instead, the number of zones is increased from 4 to 12 in the auto-altitude mode, such that the adjacent zones have the visibility difference $\Delta V = 0.5$. During the painting of the map, visibility of every pixel is found by summing its base visibility, the zone interpolation and the altitude correction, as such:

```
V = V0 + ΔV*min(X/X0,1) + acos(6371/(6371+H))
```

X is the distance from the zone contour, X_0 the mean zone width (60 for the 1st day and 120 for the 2nd), H the altitude.

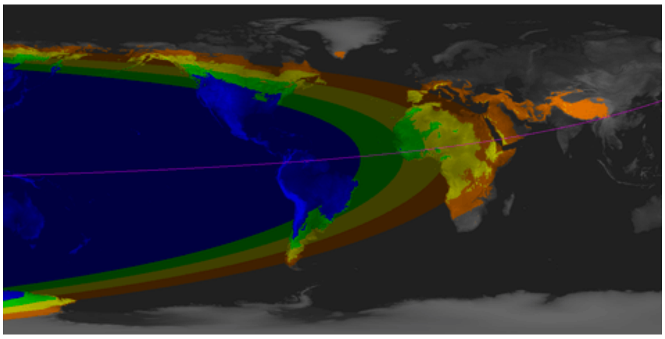

Figure 9 – Topographic Visibility Map for Safar 1440

The enhanced visibility on highland regions can easily be discriminated on the topographic map. Note that the Himalayas, which stay considerably away from the visibility border in Figure 6, are included to the visible zone after altitude correction. Another example is Figure 10, which shows the visibility on the day October 13th, 2004, when the ICOP member Jim Stamm spotted the waning crescent through an 8" telescope at 2210m elevation. This was a world record with optical aid. The location of observation near the west coast of North America (32.4°N / 110.68°W) is marked with a tiny red dot in the figure (just below the purple Best Visibility Line), which indeed lies on the visibility border for optical aid (gray/orange contour).

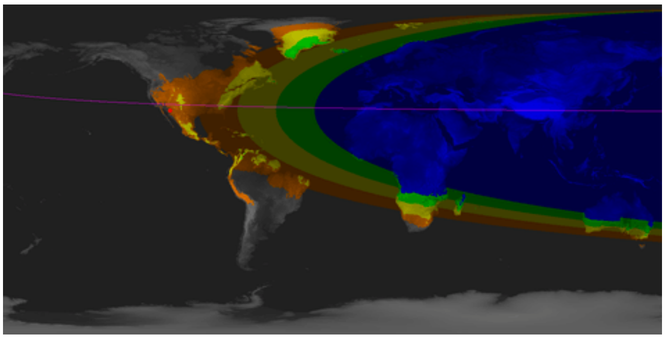

Figure 10 – Topographic Visibility Map for Sha'ban 1425

To confirm the precision of the topographic process, the visibility maps for Sha'ban 1410 are posted in Figure 11. Again, the location of John Pierce has been marked with a red dot. Although this position is evaluated as "visible by optical aid only" in the upper map, it remains within the "rarely visible by naked-eye" region in the bottom map, after the topographic correction.

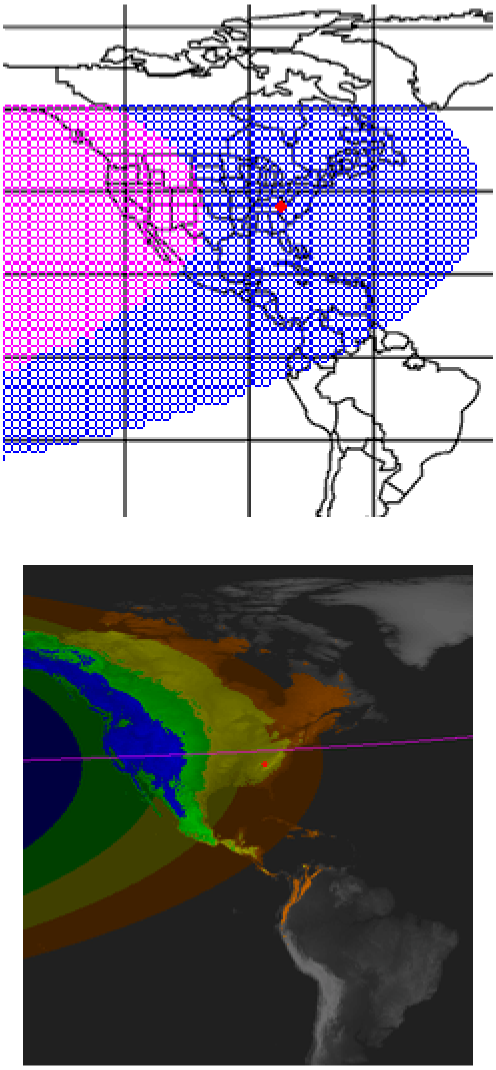

Figure 11 – Visibility Map Details for Sha'ban 1410

While Figure 10 and 11 prove the conformance of the *Extended Crescent Visibility Criterion* to observational data, verification against theoretical models would enhance its trustworthiness. The modern algorithm developed by Schaefer [14] primarily considers the two critical parameters, namely the Astronomical Extinction Coefficient and the Air-Mass. They are influenced by geographic and meteorological conditions such as altitude, humidity, temperature and aerosol content. Victor Reijs implemented Schaefer's algorithm and prepared an overlaid multiple-day visibility map of Ramadhan 1432 for Europe (Figure 12) [16]. The visibility region for the 2nd day is colored in olive, the 3rd day in violet and the 4th day in cyan. Corresponding 2nd and 3rd day maps prepared by *Urcun* are depicted in Figure 13 and Figure 14, respectively.

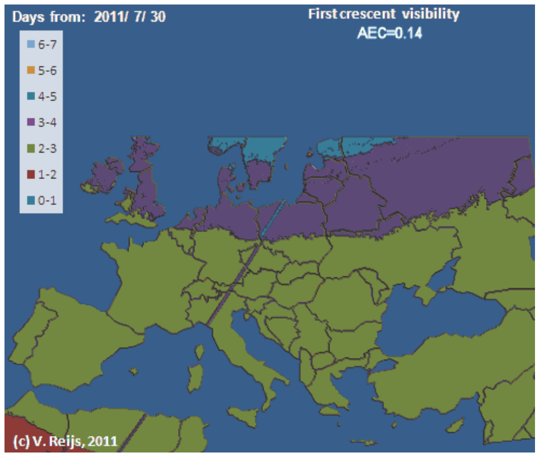

Figure 12 – Multiple-Day Visibility Map by Reijs

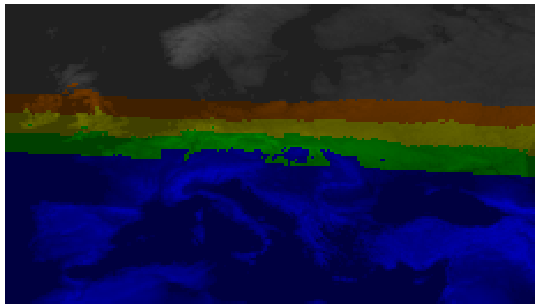

Figure 13 – 2nd Day Visibility Map by Urcun

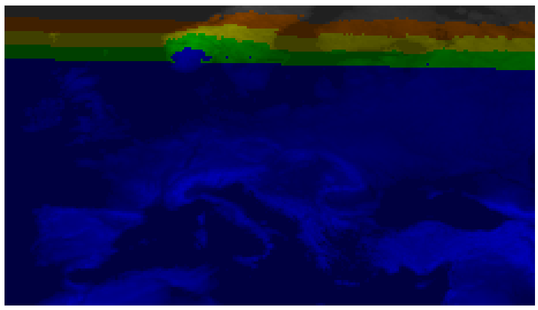

Figure 14 – 3rd Day Visibility Map by Urcun

For the 2nd day, the olive/violet border in Figure 12 should be compared with the green/yellow border (50% probability of naked-eye visibility) in Figure 13. Analogously, the violet/cyan border in Figure 12 should be checked against the green/yellow border in Figure 14 for the 3rd day. The band of difference between the maps remains below ±1° in latitude, or ca. ±0.5 in visibility, despite *Urcun* incorporates a very simple criterion. This simplicity can be recognized in the processing speed; the 400x400 map by Reijs is reported to need about one hour to complete, whilst the 1440x720 map by *Urcun* is generated in less than one second.

The crescent width is about 1.7' on the second day and 3.8' on the third. The compliance of the 2nd and 3rd day maps by *Urcun* to the Schaefer's algorithm is a proof that the *Extended Crescent Visibility Criterion* by Özlem remains valid for large crescent widths (W > 1'). A Yallop/Odeh map in contrast, includes almost the whole Europe, framed by Figure 12, into the visible region of the 2nd day.

7. CONCLUSION

The calculation of the visibility separator parabola function by a simple analytical method should shrink the computation time heavily. In order to verify the effectiveness of this method, some benchmarking is performed. Thanks to the analytical derivation, *Urcun* can complete the visibility map in less than one second. Comparison against some of the map generator software [7], namely *MoonCalc*, *Accurate Times* and *Ehille* are listed below:

	Resolution (pixels)	**Time (sec.)**	**Performance Index**
MoonCalc	360 x 180	39	0.6
Accurate Times	180 x 90	33	0.09
Ehille	1440 x 720	113 / 50	13/ 30
Urcun	1440 x 720	0.25 / 0.1	6000 / 15000

Table 1 – Benchmarking Results

The resolution of the map produced and the execution time are shown in Table 1 for each software tool. The Performance Index is computed by dividing the number of calculations into the execution time in microseconds. The number of calculations is found by multiplying the amount of total pixels by the time increments, which is assumed to be equal to the horizontal resolution. For the cases of *Ehille* and *Urcun*, most of the execution time is spent for the graphical output. Therefore both values, with and without graphics, are tabulated. As can be clearly deducted from Table 1, *Urcun* enormously outperforms all its competitors. The selection of auto-altitude mode albeit increases the execution speed from 0.25 to 0.32 sec, in any case still far less than one second.

The *Extended Crescent Visibility Criterion* chosen for the demonstration tool gives consistent results when compared to its alternatives, and the parabola created proves to be more logical throughout the Polar Regions. In that sense, the topographical application of this semi-empirical criterion with 4 parameters (W, M, S and H) can produce much proper results against the common modern 2-parameter empirical criteria, almost as accurate as the complicated theoretical model. With the analytical approach explained herein, the implementation of a very fast yet acute visibility map generator has been demonstrated.

In general, the application of analytic approach can speed up any map generation software, whereas the introduction of topography would contribute to the accuracy and reliability of crescent visibility algorithms.

8. DISCUSSION

In this section, it will first be analyzed, how a stellar visibility solution can be applied to the special case of lunar detection. A stellar model for twilight, based on a large observational dataset [19], asserts that the limiting Sun elevation can be calculated as[7]:

```
S = −2.65 − 1.23 * m − Δh
```

Here, *m* is the apparent magnitude of the celestial object, which has a logarithmic scale of base 2.5. This equation, valid for 0° > S > −7.7°, implies that the sky brightness is tripled for every degree increase of the Sun elevation. But during twilight, the luminosity alters also with the azimuth; the closer to the Sun, the brighter will be the sky. As such, *Δh* denotes the extra Sun depression, which will be necessary for the objects horizontally near to the Sun. The aforementioned model offers a linear gradient, beginning from DAZ = 58° and incrementing by 0.0338° for every degree decrease of DAZ. The model exerts this correction for up to DAZ = 0°; though it was verified for DAZ > 20°. We evaluate that the sky brightness remains nearly constant when the object is in the vicinity of the Sun; therefore we will limit this correction at DAZ = 10°, such that:

```
Δh = 0.0338 * (58 − max(DAZ, 10))
```

The model can furthermore be improved in terms of a daylight extension (S > 0°). The sky brightness loses its steepness as the Sun elevates and practically becomes flat after S > 5°. The following term is offered for this correction:

$$S_1 = S + 2^{S-2}$$

[7] The original constant was −2.47 for an extinction coefficient *k* = 0.25 and a sight height H = 0.18 km, which has been adjusted for *k* = 0 and H = 0.

Regarding the Moon, m is a function of the elongation ARCL, as follows [11]:

$$m_0 = -12.73 + 0.026*(180-\mathrm{ARCL}) + 4*10^{-9}*(180-\mathrm{ARCL})^4$$

Nevertheless, this value cannot be directly inserted into the model equation, since the lunar crescent, unlike the stars, is not a point source; it is rather a curved line. The perceived brightness of a thin crescent with W < 5’ (corner angle for the visual limit), will diminish proportional to its width. On a logarithmic magnitude scale, the following modification is hence necessary:

$$m = m_0 + 2.5 * \log(5 / W)$$

This stellar model, adapted for lunar visibility as explained above, has been tested against the simple alternative stipulated by Özlem, which is:

$$S_2 = 6 * \sqrt{W} - 4.9$$

For 0.25’ > W > 5’, the gap[8] between the model (S_1) and the Özlem’s criterion (S_2) remains within ±0.25 (Table 2).

W	S_1	S_2	Δ
0.25	-6.12	-5.90	0.22
0.5	-4.58	-4.66	-0.08
1	-2.66	-2.90	-0.24
1.5	-1.38	-1.55	-0.17
2	-0.36	-0.41	-0.05
3	1.35	1.49	0.14
4	2.97	3.10	0.13
5	4.73	4.52	-0.21

Table 2 – Comparison of the Model with the Criterion

We will next discuss the atmospheric extinction. The apparent brightness of a celestial body observed on Earth will be its brightness in the space multiplied by the extinction value. Extinction is a function of the optical air-mass and it is exponential in nature, such that its contribution can be added to the logarithmic magnitude value m:

$$m' = m + k * F(z)$$

Here, m stands for the apparent magnitude outside the atmosphere and k represents the Atmospheric Extinction Coefficient. F(z) denotes the air-mass (optical path length) quantity, which is a function the zenith angle z. In the stellar model, the following equation is chosen for F(z):

$$F(z) = 1 / (\cos(z) + 0.025 * e^{-11*\cos(z)})$$

$$z = 90.25 - M$$

The zenith angle z is complementary to the Moon elevation M. We add 0.25° because the mid-crescent is roughly one radius lower than the center of the Moon. Özlem prefers to use a shorter substitute for F(z):

$$-1.23 * k * F(z) = -0.28 / \tan(M + 1.5)$$

The both sides of the equation have the best match (±0.2) for k = 0.187. Thus, we may insert the extinction coefficient k into the Özlem’s criterion as the 5th parameter, such that[9]:

$$\begin{aligned} \mathbf{V} = & -1.5 * k / \tan(M + 1.5) \\ & - \max(\min(S, 5), -10) + 6 * \sqrt{\min(W, 5)} \\ & + \mathrm{acos}(6371/(6371+H)) - 4.9 \end{aligned}$$

This compliance proves that the “best time” indeed occurs when M = 2.5, independent of the lag time, in contrary to Yallop, which can also be confirmed by heliacal visibility arcs (Figure 15) [11].

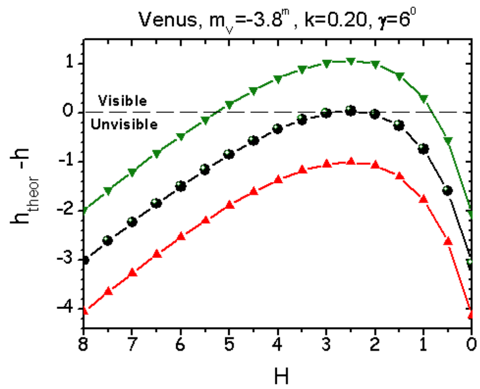

Figure 15 – Visibility Arc of Venus

However, the slope calculations disclose that the best angle of visibility for M shifts when k changes; the correlation is indicated in Table 3. The results are in accordance with Sultan’s findings, who demonstrated that the optimum Moon elevation[10] will be about 2° for k = 0.14 [17].

[8] Atmospheric extinction at M = 2.5 is accounted in both calculations.

[9] The *max* term has been included for the deep-night (no-twilight) visibility.

[10] The Atmospheric Extinction Coefficient of k = 0.14 was not declared explicitly in his paper; it has been figured out by the ratios of extra-atmospheric and ground-observed Moon luminance values.

k	0.1	0.15	0.2	0.25	0.3
Best Angle	1.4	2.1	2.6	3.1	3.6

Table 3 – Extinction Coefficient & Best Angle

When the Moon grows 20 times from W = 0.25’ to 5’, its *m* value changes from -5 to -8, implying a 16-fold increase. This means that the brightness remains roughly constant. But on the way from W = 5’ to 31’ (full moon) which is 6-fold, *m* differs by 4.7, namely 76-fold. This tells us that the full moon is 12 times brighter than the limiting thick crescent. This should result in different visibility levels between the thick crescent and a gibbous/full disc, which contradicts to the daytime observations by Özlem; he reports that the rise/set threshold elevations remain practically unchanged. Some discussion about this paradox is given below:

The perception capability of the human eye is not the same for different shapes of an object with same size and brightness. A study about the contrast thresholds of individual letters shows that the contrast sensitivity of similarly identifiable big letters differs considerably near the visual acuity limit; for identification of large letters, presence of higher object frequencies (edges) is more discriminative [20]. A crescent with sharp lips & tips will have a higher sensitivity than a soft circular full moon. Below are displayed the high & low contrast moon pictures of 5’, 20’ and 30’ width (Figure 16, left to right). Though the full moon looks much brighter than the crescent in high contrast (night), the perceptibility just equals in case of very poor contrast (daytime):

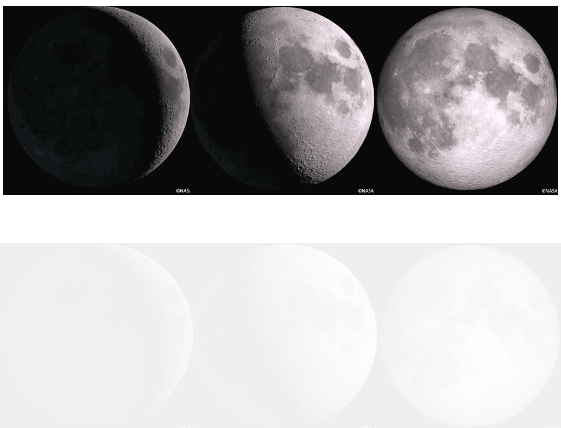

Figure 16 – High & Low Contrast Moon Views

Since the discrimination of the human eye is based on sharp contrast changes at the rims, the visibility seems to be indifferent for W > 5’. In fact, it is really not possible to hardly discern the brighter middle of a gibbous during its rise/set in daytime, without first to distinguish it at its outer circular border.

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APPENDIX

In this section are listed the simplified formulae of the astronomical parameters necessary for the calculation of the crescent visibility.

<u>SUN</u>:

Greenwich Mean Sidereal Time: $GMST = 280.461° + 360.98564737° * d$

Mean Longitude: $L0_s = 280.466° + 0.98564737° * d$

Mean Anomaly: $M_s = 357.529° + 0.98560028° * d$

True Longitude: $L_s = L0_s + 1.914° * \sin(M_s) + 0.02° * \sin(2 * M_s)$

Obliquity of the Ecliptic: $\varepsilon = 23.43929° - 0.000000356° * d$

Right Ascension: $\alpha_s = \mathrm{atan}(\sin(L_s)*\cos(\varepsilon) / \cos(L_s))$

Declination: $\delta_s = \mathrm{asin}(\sin(\varepsilon) * \sin(L_s))$

Local Hour Angle: $\omega_s = GMST - \alpha_s + \ell$

Local Azimuth: $a_s = -\mathrm{atan}(\sin(\omega_s) / (\cos(\varphi)*\tan(\delta_s)-\sin(\varphi)*\cos(\omega_s)))$

Local Elevation: $e_s = \sin(\cos(\omega_s)*\cos(\delta_s)*\cos(\varphi)+\sin(\delta_s)*\sin(\varphi))$

<u>MOON</u>:

Mean Elongation: $D = 117.850° + 12.19074911° * d$

Mean Anomaly: $M_m = 134.963° + 13.06499295° * d$

Argument of Latitude: $F = 93.272° + 13.22935024° * d$

Ecliptic Longitude:

$$
\begin{aligned}
L_m = 218.316° &+ 13.17639647° * d \\
&+ 6.2888° * \sin(M_m) \\
&+ 1.2740° * \sin(2 * D - M_m) \\
&+ 0.6583° * \sin(2 * D) \\
&+ 0.2136° * \sin(2 * M_m) \\
&- 0.1851° * \sin(M_s) \\
&- 0.1143° * \sin(2 * F) \\
&+ 0.0588° * \sin(2 * D - 2 * M_m) \\
&+ 0.0571° * \sin(2 * D - M_s - M_m) \\
&+ 0.0533° * \sin(2 * D + M_m) \\
&+ 0.0458° * \sin(2 * D - M_s) \\
&- 0.0409° * \sin(M_s - M_m) \\
&- 0.0347° * \sin(D) \\
&- 0.0304° * \sin(M_s + M_m) \\
&+ 0.0153° * \sin(2 * D - 2 * F) \\
&- 0.0125° * \sin(M_m + 2 * F) \\
&+ 0.0110° * \sin(M_m - 2 * F)
\end{aligned}
$$

Ecliptic Latitude:

$$
\begin{aligned}
\beta_m = &+ 5.1281° * \sin(F) \\
&+ 0.2806° * \sin(M_m + F) \\
&+ 0.2777° * \sin(M_m - F) \\
&+ 0.1732° * \sin(2 * D - F) \\
&+ 0.0554° * \sin(2 * D - M_m + F) \\
&+ 0.0463° * \sin(2 * D - M_m - F) \\
&+ 0.0326° * \sin(2 * D + F)
\end{aligned}
$$

Geocentric Distance [Mm]:	$R0$	$= 385.0006 - 20.9054 * \cos(M_m) - 3.6991 * \cos(2 * D - M_m) - 2.9560 * \cos(2 * D) - 0.5699 * \cos(2 * M_m) + 0.2462 * \cos(2 * D - 2 * M_m) - 0.2046 * \cos(2 * D - M_s) - 0.1707 * \cos(2 * D + M_m) - 0.1521 * \cos(2 * D - M_s - M_m) - 0.1296 * \cos(M_s - M_m) + 0.1087 * \cos(D) + 0.1048 * \cos(M_s + M_m) + 0.0797 * \cos(M_m - 2 * F) + 0.0489 * \cos(M_s)$
Topocentric Distance [Mm]:	R	$= R0 - 6.371 * \sin(e_m)$
Right Ascension:	α_m	$= \mathrm{atan}((\sin(L_m)*\cos(\varepsilon)-\tan(\beta_m)*\sin(\varepsilon)) / \cos(L_m))$
Declination:	δ_m	$= \mathrm{asin}(\sin(\beta_m)*\cos(\varepsilon) + \cos(\beta_m)*\sin(\varepsilon)*\sin(L_m))$
Local Hour Angle:	ω_m	$= GMST - \alpha_m + \ell$
Local Azimuth:	a_m	$= -\mathrm{atan}(\sin(\omega_m) / (\cos(\varphi)*\tan(\delta_m)-\sin(\varphi)*\cos(\omega_m)))$
Local Elevation:	e_m	$= \sin(\cos(\omega_m)*\cos(\delta_m)*\cos(\varphi)+\sin(\delta_m)*\sin(\varphi))$
Elevation corrected for Parallax [°]:	e_m'	$= e_m - \cos(e_m) * 365 / R$
Phase:	P	$= L_m - L_s$
Arc of Light:	$ARCL$	$= \mathrm{acos}(\cos(P) * \cos(\beta_m))$
Illumination:	I	$= (1 - \cos(ARCL)) / 2$
Crescent Width [arc-min]:	W	$= 11950 * I / R;$

Where:

d:	Time elapsed (in days) since 01/01/2000, 12:00 GMT
φ:	Latitude of location
ℓ:	Longitude of location